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1993 J. Phys. A: Math. Gen. 26 3285

(http://iopscience.iop.org/0305-4470/26/13/030)

J. Phys. A: Math. Gen. 26 (1993) 3285–3290. Printed in the UK

How to obtain the quantum state of a free-electron laser with an axial-guide magnetic field from its classical trajectories

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Received 24 July 1992, in final form 4 January 1993

Abstract. A procedure is given to obtain the quantum state of a free-electron laser with an axial-guide field from a coherent state. It is shown that the coherent state corresponds to the classical trajectories. The small-signal gain is obtained and the effects of the guide field on the gain are bounded in order to keep the orbital and electrostatic stabilities.

1. Introduction

The possible use of relativistic charged particle in devices with a spatial periodic electric or magnetic field for the generation of microwaves was first expressed by Ginzburg [1]. This idea was then developed by Motz [2] and Landeker [3] and the stimulated Bremsstrahlung by a relativistic electron beam in a wiggler field is now called free-electron lasing.

The free-electron laser (FEL) is capable of producing radiation from the ultraviolet to the microwave region of the electromagnetic spectrum and the coherent state for a wiggler-pumped FEL has been discussed in previous literature [4, 5]. Especially, in [4], it was shown that the coherent state corresponds to the classical trajectories. However, an axial-guide magnetic field is usually present in a FEL for beam confinement and gain enhancement. In this case, the treatment must be three-dimensional. A large number of works has been done on a wiggler-pumped FEL with an axial field in classical theories [6–11]. Also, in quantum theories, the equilibrium solution of the Dirac equation was given [12] and the photon statistics and squeezing properties were discussed in the weak coupling limit [13] for this configuration. In this paper, extending the method proposed in [4], we give a procedure to obtain the quantum state of a FEL with a guide-field from its classical trajectories. First, we divide the FEL system into two parts governed by the Hamiltonians H_a and H_b, respectively. The coherent state $\|\psi_{(t)}\rangle$ of the part H_b corresponds to the classical trajectories of the FEL. Then the small-signal gain of the FEL system is obtained in the classical limit. It is shown that the guide-field effects are confined for both group $-I$ and $-II$ orbits [14, 15]. Finally, the quantum state $|\Phi_{(t)}\rangle$ of the FEL system is obtained through a transformation from the coherent state $\|\psi_{(t)}\rangle$.

The following analysis is developed in the Bambini–Renieri reference frame which is obtained through a Lorentz transformation from Laboratory frame. In the Bambini-Renieri frame, the wiggler and laser fields oscillate with the same frequency and the electron beam is non-relativistic.

0305-4470/93/000000+06$07.50

2. The coherent state $\|\psi_{(t)}\rangle$ of the part H_b and the FEL gain

The Bambini–Renieri Hamiltonian for a wiggler-pumped FEL with an axial-guide magnetic field has the form [13]

$$\begin{aligned}\hat{H}=mc^2+\tfrac{1}{2}\hbar\omega_c+\hbar\omega+\frac{\hat{p}_z^2}{2m}+\hbar\omega_c\hat{T}_+\hat{T}_-\\ +\hbar\omega(a_w^+a_w+a_r^+a_r)+\hbar\Omega[a_w^+a_r\,\mathrm{e}^{2ikz}+a_wa_r^+\,\mathrm{e}^{-2ikz}]\\ -\hbar(\omega_c\Omega)^{1/2}\begin{bmatrix}a_w^+\hat{T}_-\,\mathrm{e}^{ik_1z}+a_w\hat{T}_+\,\mathrm{e}^{-ik_1z}\\ +a_r^+\hat{T}_-\,\mathrm{e}^{-ik_2z}+a_r\hat{T}_+\,\mathrm{e}^{ik_2z}\end{bmatrix}\end{aligned} \tag{1}$$

where $\hbar$ is Planck constant, c is the speed of light in vacuum, $e(>0)$ is the electron charge, m is the electron mass, $\boldsymbol{r}=(x,y,z)$ is the electron position, $\boldsymbol{p}$ is the electron momentum, V is the interaction volume, B_0 is the amplitude of the axial-guide magnetic field, $\omega_c=eB_0/m$ is the cyclotron frequency, $k_1(k_2)$ is the wavenumber of the wiggler (laser), $2k=k_1+k_2$, $\omega=V_{\mathrm{BR}}k_1=ck_2$ are the frequencies of the wiggler and laser fields with V_{BR} the velocity of Bambini–Renieri frame respecting to Laboratory frame, $\Omega=e^2/2m\omega\varepsilon_0V$ is the coupling constant with ε_0 the dielectric constant of free space, $a_w^+(a_r^+)$ and $a_w(a_r)$ are the creation and annihilation operators of the wiggler (laser), and

$$\hat{T}_\pm=\frac{(\hat{p}_x+(m\omega_c/2)y)\mp i(\hat{p}_y-(m\omega_c/2)x)}{(2m\hbar\omega_c)^{1/2}} \tag{2}$$

are the creation and annihilation operators of the electron vertical cyclotron motion. The eigenvalue problems for the number operators are

$$a_w^+a_w|n_w\rangle=n_w|n_w\rangle \tag{3}$$

$$a_r^+a_r|n_r\rangle=n_r|n_r\rangle \tag{4}$$

$$\hat{T}_+\hat{T}_-|n_T\rangle=n_T|n_T\rangle \tag{5}$$

with n_w, n_r and $n_T=0,1,2,\ldots$. From (1), one can obtain three constants of motion

$$x\hat{p}_y-y\hat{p}_x+\hbar\hat{T}_+\hat{T}_-=\text{constant }\hat{L} \tag{6}$$

$$a_w^+a_w+a_r^+a_r+\hat{T}_+\hat{T}_-=\text{constant }\hat{M} \tag{7}$$

$$\hat{p}_z+2\hbar ka_r^+a_r+\hbar k_1\hat{T}_+\hat{T}_-=\text{constant }\hat{N}. \tag{8}$$

The above three formulae are the conservation laws of the angular momentum, quanta number, and linear momentum of the FEL system, respectively. If we define the combined operators

$$\hat{A}=a_r\,\mathrm{e}^{2ikz} \tag{9}$$

$$\hat{B}=\hat{T}_-\,\mathrm{e}^{ik_1z} \tag{10}$$

then the Hamiltonian (1) converts into

$$H=mc^2+\hbar\omega+\tfrac{1}{2}\hbar\omega_c+H_a+H_b \tag{11}$$

where

$$H_a=\frac{\hat{p}_z^2}{2m}+\hbar\omega a_w^+a_w+\left[\hbar\omega+\frac{2\hbar k}{m}\langle\hat{p}_z\rangle\right]\hat{A}^+\hat{A}+\left[\hbar\omega+\frac{\hbar k_1}{m}\langle\hat{p}_z\rangle\right]\hat{B}^+\hat{B} \tag{12}$$

$$\begin{aligned}H_b=-\frac{2\hbar k}{m}\langle\hat{p}_z\rangle\hat{A}^+\hat{A}+\left[\hbar(\omega_c-\omega)-\frac{\hbar k_1}{m}\langle\hat{p}_z\rangle\right]\hat{B}^+\hat{B}+\hbar\Omega(a_w^+\hat{A}+a_w\hat{A}^+)\\ -\hbar(\omega_c\Omega)^{1/2}[a_w^+\hat{B}+a_w\hat{B}^++\hat{A}^+\hat{B}+A\hat{B}^+].\end{aligned} \tag{13}$$

The formula (11) represents the Hamiltonian of three coupled harmonic oscillators described by a_w, $\hat{A}$ and $\hat{B}$. The operators a_w, $\hat{A}$, and $\hat{B}$ have the following coherent states, respectively

$$|\alpha_{w(t)}\rangle = e^{-|\alpha_{w(t)}|^2/2} \sum_{n_w=0}^{\infty} \frac{\alpha_{w(t)}^{n_w}}{\sqrt{n_w!}} |n_w\rangle \tag{14}$$

$$\|\alpha_{(t)}\rangle = e^{-|\alpha_{(t)}|^2/2} \sum_{n_r=0}^{\infty} e^{-2in_r kz} \frac{\alpha_{(t)}^{n_r}}{\sqrt{n_r!}} |n_r\rangle \tag{15}$$

$$\|\beta_{(t)}\rangle = e^{-|\beta_{(t)}|^2/2} \sum_{n_T=0}^{\infty} e^{-in_T k_1 z} \frac{\beta_{(t)}^{n_T}}{\sqrt{n_T!}} |n_T\rangle \tag{16}$$

with

$$a_w|\alpha_{w(t)}\rangle = \alpha_{w(t)}|\alpha_{w(t)}\rangle \tag{17}$$

$$\hat{A}\|\alpha_{(t)}\rangle = \alpha_{(t)}\|\alpha_{(t)}\rangle \tag{18}$$

$$\hat{B}\|\beta_{(t)}\rangle = \beta_{(t)}\|\beta_{(t)}\rangle. \tag{19}$$

It is obvious that the Hamiltonian (11) does not preserve the coherent state $\|\psi_{(t)}\rangle = e^{i\theta_{(t)}}\|\alpha_{(t)}\rangle\|\beta_{(t)}\rangle|\alpha_{w(t)}\rangle$ because of the quadratic nonlinearity term $\hat{p}_z^2/2m = [\hat{N} - 2\hbar k\hat{A}^+\hat{A} - \hbar k_1 \hat{B}^+\hat{B}]^2/2m$. However, the Hamiltonian H_b preserves the coherent state $\|\psi_{(t)}\rangle$. Starting from the initial coherent state $\|\psi_0\rangle = e^{i\theta_0}\|\alpha_0\rangle\|\beta_0\rangle|\alpha_{w0}\rangle$, a system governed only by H_b will be in $\|\psi_{(t)}\rangle$ at time t where $\alpha_{(t)}$, $\beta_{(t)}$, $\alpha_{w(t)}$ and $\theta_{(t)}$ satisfy

$$\frac{d\alpha_{(t)}}{dt} = \frac{2ik\langle\hat{p}_z\rangle}{m}\alpha_{(t)} + i(\omega_c\Omega)^{1/2}\beta_{(t)} - i\Omega\alpha_{w(t)} \tag{20}$$

$$\frac{d\beta_{(t)}}{dt} = \left[\frac{ik_1\langle\hat{p}_z\rangle}{m} + i(\omega - \omega_c)\right]\beta_{(t)} + i(\omega_c\Omega)^{1/2}(\alpha_{(t)} + \alpha_{w(t)}) \tag{21}$$

$$\frac{d\alpha_{w(t)}}{dt} = -i\Omega\alpha_{(t)} + i(\omega_c\Omega)^{1/2}\beta_{(t)} \tag{22}$$

$$\frac{d\theta_{(t)}}{dt} = -\frac{i}{2}\frac{d}{dt}[|\alpha_{(t)}|^2 + |\beta_{(t)}|^2 + |\alpha_{w(t)}|^2] \tag{23}$$

with the initial conditions $\alpha_{(0)} = \alpha_0$, $\beta_{(0)} = \beta_0$, $\alpha_{w(0)} = \alpha_{w0}$ and $\theta_{(0)} = \theta_0$. Equations (20)–(23) are obtained by inserting the coherent state $\|\psi_{(t)}\rangle$ and the sub-Hamiltonian (13) into Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\|\psi_{(t)}\rangle = H_b\|\psi_{(t)}\rangle. \tag{24}$$

Also, one can consider the wiggler as a c-constant-number (that is $\alpha_{w(t)}$ = constant α_{w0}), since it is intense compared with the radiation. Further, we define the classical amplitude $\tilde{\alpha}_{r(t)}$ and relative phase θ_r for the radiation field, the classical momentum $p_z(t)$, classical position $z_{(t)}$, classical cyclotron radius $r_{(t)}$, and classical cyclotron phase $\phi_{(t)}$ for the electron as follows

$$p_{z(t)} = \langle\hat{p}_z\rangle = m\frac{dz_{(t)}}{dt} \tag{25}$$

$$r_{(t)}\, e^{i\phi_{(t)}} = \beta_{(t)}\, e^{-i(k_1 z_{(t)} + \omega t)} \tag{26}$$

$$\tilde{\alpha}_{r(t)}\, e^{i\theta_r} = \alpha_{(t)}\, e^{-2ikz_{(t)}}. \tag{27}$$

Then (20) and (21) give the classical equations for the electron motion [10]

$$\frac{\mathrm{d}\beta_{(t)}}{\mathrm{d}t}=\mathrm{i}\left[\omega-\omega_c+k_1\frac{\mathrm{d}z_{(t)}}{\mathrm{d}t}\right]\beta_{(t)}+\mathrm{i}(\omega_c\Omega)^{1/2}[\alpha_{w0}+\tilde{\alpha}_{r(t)}\,\mathrm{e}^{\mathrm{i}(2kz_{(t)}+\theta_r)}] \tag{28}$$

$$\begin{aligned}\frac{\mathrm{d}^2z_{(t)}}{\mathrm{d}t^2}=&-\frac{2\hbar k_1\alpha_{w0}(\omega_c\Omega)^{1/2}}{m}[\mathrm{Im}\,\beta_{(t)}]\\&+\frac{2\hbar k_2\tilde{\alpha}_{r(t)}(\omega_c\Omega)^{1/2}}{m}\{\mathrm{Im}[\beta_{(t)}\,\mathrm{e}^{-\mathrm{i}(2kz_{(t)}+\theta_r)}]\}\\&+\frac{4\hbar k\Omega\alpha_{w0}\tilde{\alpha}_{r(t)}}{m}\sin(2kz_{(t)}+\theta_r).\end{aligned} \tag{29}$$

The equations (28) and (29) can be solved in the small-signal regime, where each variable $G=\bar{G}+\tilde{G}$ with $\bar{G}$ the equilibrium solution in the absence of the radiation and $\tilde{G}$ the perturbative solution in the first order of the radiation. One can obtain

$$\bar{z}_{(t)}=V_0(t-t_0) \tag{30}$$

$$\bar{\beta}=\alpha_{w0}(\omega_c\Omega)^{1/2}/(\omega_c-\omega-V_0k_1) \tag{31}$$

where V_0 is the equilibrium axial velocity of the electron and t_0 is the initial time at which the electron enters the wiggler ($\bar{z}=0$ plane). For the weak coupling case $\Omega\to 0$ with $\Omega\alpha_{w0}=constant$, *the equation for the axial bunching is obtained from* (28) *and* (29)

$$\frac{\mathrm{d}^2\tilde{z}_{(t)}}{\mathrm{d}t^2}=\frac{4\hbar k\Omega\alpha_{w0}\tilde{\alpha}_{r(t)}}{m}\left[1-\frac{\omega_c(\omega_c-\omega-V_0k_1+V_0k_2)}{(\omega_c-\omega-V_0k_1)(\omega_c-\omega+V_0k_2)}\right]\sin(\Delta\omega t+\Psi_0) \tag{32}$$

where $\Delta\omega=2V_0k$ is the deturning parameter and $\Psi_0=\theta_r-\Delta\omega t_0$ is the initial phase. Under the initial conditions $\tilde{z}_{(0)}=0$ and $(\mathrm{d}/\mathrm{d}t)\tilde{z}_{(0)}=0$, the solution of (32) is

$$\begin{aligned}\tilde{z}_{(t)}=&-\frac{\hbar\Omega\alpha_{w0}\tilde{\alpha}_{r(t)}}{mkV_0^2}\left[1-\frac{\omega_c(\omega_c-\omega-V_0k_1+V_0k_2)}{(\omega_c-\omega-V_0k_1)(\omega_c-\omega+V_0k_2)}\right]\\&\times[\sin(\Delta\omega t+\Psi_0)-\Delta\omega t\cos\Psi_0-\sin\Psi_0].\end{aligned} \tag{33}$$

The equations (20) and (21) also give the Colson–Maxwell equation [16]

$$\frac{\mathrm{d}\alpha_{r(t)}}{\mathrm{d}t}=\frac{\Omega\alpha_{w0}(\omega+V_0k_1)}{(\omega_c-\omega-V_0k_1)}\langle\sin(2kz_{(t)}+\theta_r)\rangle \tag{34}$$

where $\langle\ldots\rangle=(1/2\pi)\int_0^{2\pi}\mathrm{d}\Psi_0(\ldots)$ means the average for different electrons. Substituting (30) and (33) into (34) gives the gain on the interaction time T

$$G_{T(B_0)}=\int_0^T\frac{\mathrm{d}\tilde{\alpha}_{r(t)}}{\tilde{\alpha}_{r(t)}}=\eta G_{T(B_0=0)} \tag{35}$$

where $G_{T(B_0=0)}$ is the gain of a wiggler-pumped FEL [17], and

$$\eta=\left[1-\frac{\omega_c(\omega_c-\omega-V_0k_1+V_0k_2)}{(\omega_c-\omega-V_0k_1)(\omega_c-\omega+V_0k_2)}\right]\Big/\left[I-\frac{\omega_c}{(\omega+V_0k_1)}\right] \tag{36}$$

describes the effects of the axial-guide magnetic field. For conventional FEL parameters, $|\omega-\omega_c|\gg V_0K$, so we have

$$\eta=1\Big/\left[1-\frac{eB_0}{m\gamma_0V_{z0}k_w}\right]^2 \tag{37}$$

in the Laboratory frame, where γ_0 is the relativistic factor, V_{z0} is the axial velocity of the electron beam, and K_w is the wavenumber of the wiggler. Using the orbital [10, 18] and the electrostatic [19] stability conditions, one can obtain [20]

$$\eta \leqslant [1-\gamma_0^{-2}]^{2/3}\beta_w^{-4/3} = \eta_I \tag{38}$$

for group-I orbits [14–15] characterized by $\omega_c < \omega$ and

$$\eta \leqslant [\gamma_0\beta_w]^{-4/3} = \eta_{II} \tag{39}$$

for group-II orbits [14–15] characterized by $\omega_c > \omega$, where $\beta_w = eB_w/m\gamma_0 cK_w$, and B_w is the amplitude of the wiggler field. So the amplifying effects of the axial-field on the gain are limited. For $\beta_w = 0.05$ and $\gamma_0 = 3.94$, we have $\eta_I = 51.9$ and $\eta_{II} = 8.7$. In general, $\gamma_0 \gg 1$, then $\eta_I/\eta_{II} \simeq \gamma_0^{4/3} \gg 1$. The FEL device must be operated in the group-I orbits in order to give the maximum amplifying effects of the axial guide field.

3. The quantum state of the FEL system

If the FEL system is in the coherent state $\|\psi_0\rangle$ initially, the quantum state $|\Phi_{(t)}\rangle$ of the FEL system is not the coherent state $\|\psi_{(t)}\rangle$ at time t. However, we can obtain $|\Phi_{(t)}\rangle$ from $\|\psi_{(t)}\rangle$ through a unitary transformation

$$|\Phi_{(t)}\rangle = S_{(t)}\|\psi_{(t)}\rangle \tag{40}$$

$$S_{(t)}^{+}S_{(t)} = 1 \tag{41}$$

where the transformation operator $S_{(t)}$ satisfy

$$\mathrm{i}\hbar(\mathrm{d}S_{(t)}/\mathrm{d}t) = H_a S_{(t)} + [H_b, S_{(t)}] \tag{42}$$

with the initial condition $S_{(0)} = 1$. The solution of (42) is

$$\begin{aligned} S_{(t)} = 1 &+ \frac{1}{\mathrm{i}\hbar}\int_0^t \mathrm{d}t_1\, H_{a(t_1)} + \frac{1}{(\mathrm{i}\hbar)^2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \{H_{a(t_1)}H_{a(t_2)} + [H_{b(t_1)}, H_{a(t_2)}]\} \\ &+ \frac{1}{(\mathrm{i}\hbar)^3}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \\ &\times \begin{Bmatrix} H_{a(t_1)}H_{a(t_2)}H_{a(t_3)} + H_{a(t_1)}[H_{b(t_2)}, H_{a(t_3)}] \\ +[H_{b(t_1)}, H_{a(t_2)}H_{a(t_3)}] + [H_{b(t_1)}, [H_{b(t_2)}, H_{a(t_3)}]] \end{Bmatrix} \\ &+ \ldots \end{aligned} \tag{43}$$

In general, $|\Phi_{(t)}\rangle$ is not a coherent state and the procedure to obtain the expression of $|\Phi_{(t)}\rangle$ is tedious. However, under the classical assumption $\hat{p}_z^2 = 2\langle\hat{p}_z\rangle\hat{p}_z - \langle\hat{p}_z\rangle^2$, one can obtain $[H_{a(t_1)}, H_{b(t_2)}] = 0$ and $[S_{(t)}, \hat{A}^+\hat{A}] = 0$, so $|\Phi_{(t)}\rangle$ is a coherent state. This is just the result of [4].

In conclusion, once the classical trajectories of the FEL system are given, a coherent state $\|\psi_{(t)}\rangle$ can be formulated through the formulae (25)–(27). The unitary operator $S_{(t)}$ can be obtained by using the equation (43). Then the quantum state $|\Phi_{(t)}\rangle$ of the FEL system can be obtained from the formula (40).

Acknowledgment

This work was partially supported by the Chinese National Science Foundation.

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